Phase-field approximation of cohesive fracture energies

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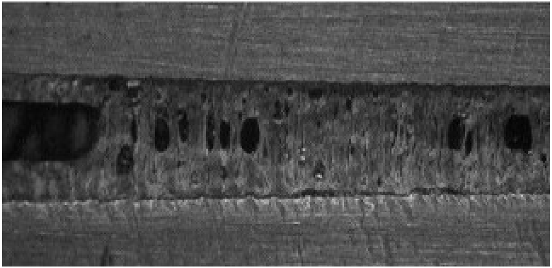
Introduction

Consider an elastic body subjected to forces and boundary conditions.

Cohesive models: fracture energy is gradually released with the growth of the crack opening (progressive weakening of the bound between the lips)



Failure process of a porous adhesive joint in opening mode fracture [Zhu-Liechti-Ravi-Chandar '09]

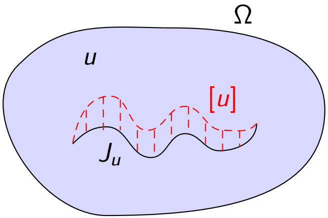
[Del Piero-Truskinovsky '98, Francfort-Marigo '98, Braides-Dal Maso-Garroni '99, Alicandro-Braides-Shah '99, Alicandro- Focardi '01, Dal Maso-Zanini '07, Bourdin-Francfort-Marigo '08, Dal Maso-Garroni '08, Cagnetti-Toader '11, Larsen-Slastikov '14, Conti-Focardi-Iurlano '16, Dal Maso-Orlando-Toader '16, Crismale-Lazzaroni-Orlando '16, Larsen-Li '16, Negri-Scala '17 & '20, Thomas-Zanini '17, Negri-Vitali '18, Bonacini-Conti-Iurlano '21,. . .]

Cohesive fracture: scalar sharp model

Static scalar cohesive energy [Dugdale '60, Barenblatt '62]:

$$F(u) := \underbrace{\int_{\Omega \setminus J_u} |\nabla u|^2 \mathrm{d}x}_{\text{stored elastic energy}} + \underbrace{\int_{J_u} g_{\mathrm{scal}}(|[u]|) \mathrm{d}\mathcal{H}^{d-1}}_{\text{crack energy}}, \qquad u \in SBV(\Omega),$$

$\Omega \subset \mathbb{R}^d$ reference configuration
$J_u \subset \Omega$ crack, (d-1)-dimensional
$u : \Omega \setminus J_u \to \mathbb{R}$ elastic displacement
$[u] : J_u \to \mathbb{R}$ amplitude of the crack



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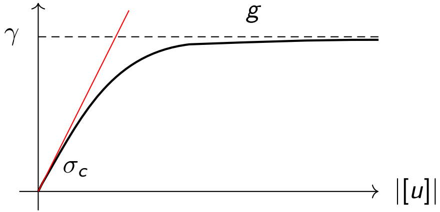

with $\sigma_c := g'_{\mathrm{scal}}(0) \in (0, +\infty)$ and $\gamma := g_{\mathrm{scal}}(+\infty)$.

Relaxed formulation

We will deal with global minima, but F does not attain its infimum.
Introduce the L^1-**relaxation** F_{scal} of F (defined on $GBV(\Omega)$):

$$F_{\mathrm{scal}}(u) := \int_\Omega h_{\mathrm{scal}}(|\nabla u|)\,\mathrm{d}x + \int_{J_u} g_{\mathrm{scal}}(|[u]|)\,\mathrm{d}\mathcal{H}^{d-1} + \sigma_c |D^c u|(\Omega)$$

with $h_{\mathrm{scal}}(s) := (s^2 \wedge \sigma_c s)^{\mathrm{conv}}$; g_{scal} and σ_c are as before.

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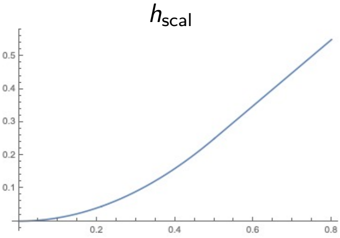

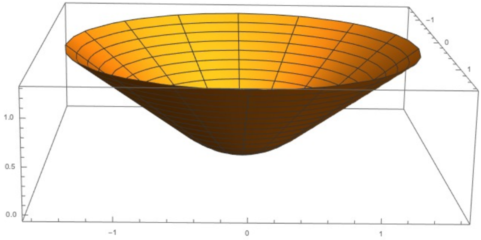

Brittle fracture: scalar sharp and phase-field models

Ambrosio-Tortorelli '92: for $u, v \in H^1(\Omega)$, $0 \le v \le 1$, consider

$$F_\delta^{AT}(u,v) := \int_\Omega \left(v^2 |\nabla u|^2 + \gamma^2 \left(\frac{(1-v)^2}{4\delta} + \delta |\nabla v|^2 \right) \right) \mathrm{d}x.$$

As $\delta \to 0$, AT_δ Γ-converges in $L^1 \times L^1$ to

$$MS(u) := \int_\Omega |\nabla u|^2 dx + \gamma \mathcal{H}^{d-1}(J_u),$$

if $u \in GSBV(\Omega)$ and $v = 1$ a.e..

Convergence of minimum problems hold.

Numerical simulations: [Bourdin-Francfort-Marigo '00, Bourdin '07, Del Piero-Lancioni-March '07, Negri '07, Amor-Marigo-Maurini '09, Lancioni-Royer Carfagni '09, Freddi-Royer Carfagni '10, Schmidt-Fraternali-Ortiz '11, Bourdin-Larsen-Richardson '11, Borden-Verhoosel-Scott '12, Ambati-Gerasimov-De Lorenzis '15,. . .]

Brittle vs cohesive: scalar phase-field models

Ambrosio-Tortorelli '92 model: $\delta > 0$ small, toughness γ

$$F_\delta^{AT}(u,v) := \int_\Omega \left(v^2 |\nabla u|^2 + \gamma^2 \left(\frac{(1-v)^2}{4\delta} + \delta |\nabla v|^2 \right) \right) \mathrm{d}x$$

Conti-Focardi-F.I. '16 model: $\delta > 0$ small, toughness γ, critical stress σ_c

$$F_\delta^{\mathrm{scal}}(u,v) := \int_\Omega \left(f_\delta(v)^2 |\nabla u|^2 + B \frac{(1-v)^2}{4\delta} + C\delta |\nabla v|^2 \right) \mathrm{d}x.$$

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$$f_\delta(v)^2 := \underbrace{1}_{\text{elastic phase}} \wedge \underbrace{\frac{A\delta\, v^2}{(1-v)^2}}_{\text{pre-fracture phase}}, \quad \gamma := \sqrt{BC}, \quad \sigma_c := \sqrt{AB}.$$

Convergence to the cohesive sharp model

Theorem (Conti-Focardi-F.I. '16). For $u, v \in H^1(\Omega)$, $0 \le v \le 1$, let

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$$F_{\mathrm{scal}}(u) := \int_\Omega h_{\mathrm{scal}}(|\nabla u|)\mathrm{d}x + \int_{J_u} g_{\mathrm{scal}}(|[u]|)\mathrm{d}\mathcal{H}^{d-1} + \sigma_c |D^c u|(\Omega),$$

if $u \in GBV(\Omega)$ and $v = 1$ a.e.; $h_{\mathrm{scal}}(s) := (s^2 \wedge \sigma_c s)^{\mathrm{conv}}$ for $s \in \mathbb{R}_+$;
g_{scal} is given by a cell formula (see later).

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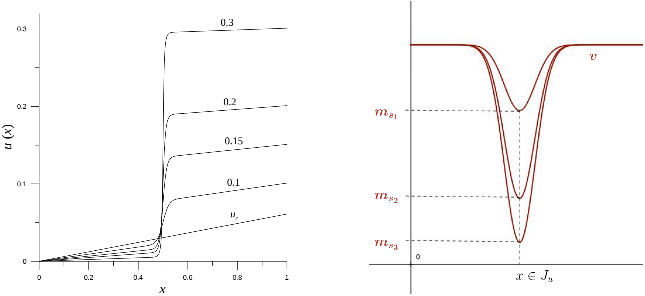
g_{scal} is given by a cell formula (see later).

Convergence of minimum points and values holds (adding a penalization or boundary conditions).

[Alternative approaches: Alicandro-Braides-Shah '99, Dal Maso-Orlando-Toader '16]

Numerical tests: the displacement

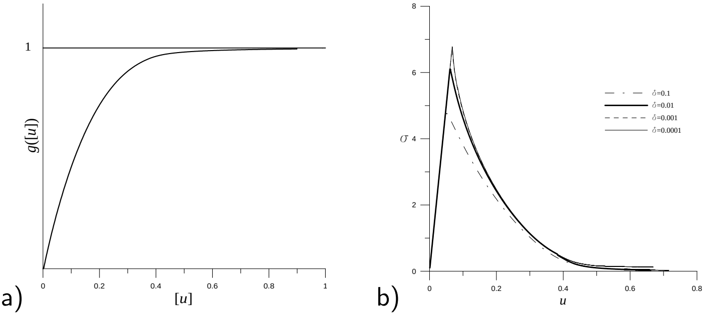
1D tests [Freddi-F.I. '16] when $\Omega = [0, 1]$, $A = B = C = 1$.



Bar in traction. Distributions of displacement u and of damage v along the bar for different values of the imposed displacement, in the case $f_\delta(v) := \min\{1, \delta v^2/(1-v)^2\}$.

Numerical tests: the surface energy

1D tests [Freddi-F.I. '16] when $\Omega = [0, 1]$, $A = B = C = 1$.



Bar in traction. Left: graph of the cohesive energy density $g_{\text{scal}}([u])$. Right: stress σ as a function of the imposed boundary displacement, for different values of the small parameter δ.

Vectorial phase-field models: isotropic potential

The simplest model [Conti-Focardi-F.I., in preparation]:

for $(u, v) \in H^1(\Omega, \mathbb{R}^m \times [0,1])$, let

$$\mathcal{F}_\delta(u,v) := \int_\Omega \left(f_\delta(v)^2 \, |\nabla u|^2 + \frac{(1-v)^2}{4\delta} + \delta |\nabla v|^2 \right) \mathrm{d}x,$$

recalling that (with $A = B = C = 1$)

$$f_\delta(v)^2 := 1 \wedge \delta f(v)^2, \qquad f(v)^2 := \frac{v^2}{(1-v)^2}.$$

Convergence to the sharp model

Theorem (Conti-Focardi-F.I. in preparation). It holds

$$\Gamma(L^1)\text{-}\lim_{\delta\to 0} \mathcal{F}_\delta(u,v) = \mathcal{F}(u,v), \quad (u,v) \in L^1(\Omega, \mathbb{R}^{m+1}),$$

where

$$\mathcal{F}(u,v) := \int_\Omega h^{qc}(\nabla u)\,\mathrm{d}x + \int_{J_u} g_{\mathrm{scal}}(|[u]|)\,\mathrm{d}\mathcal{H}^{d-1} + |D^c u|(\Omega),$$

if $u \in (GBV \cap L^1(\Omega))^m$ and $v = 1$ a.e..
Here,

$$h(\xi) := |\xi|^2 \wedge |\xi|, \quad \text{for } \xi \in \mathbb{R}^{m\times d}$$

$$h^{\mathrm{qc}}(\xi) := \inf\{\int_{(0,1)^d} h(\xi + \nabla\varphi)\mathrm{d}x \colon \varphi \in C_c^\infty((0,1)^d, \mathbb{R}^m)\}.$$

Convergence to the sharp model

Theorem (Conti-Focardi-F.I. in preparation). It holds

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if $u\in(GBV\cap L^1(\Omega))^m$ and $v=1$ a.e..
Here,

$$h(\xi):=|\xi|^2\wedge|\xi|,\quad \text{for } \xi\in\mathbb{R}^{m\times d}$$

It can be proved that h^{qc} is not convex:

$$\begin{cases} h^{\mathrm{qc}}(\xi)=h_{\mathrm{scal}}(|\xi|), & \text{if } |\xi|\le\frac{1}{2} \text{ or } \operatorname{rank}\xi\le 1;\\ h_{\mathrm{scal}}(|\xi|)<h^{\mathrm{qc}}(\xi)\le|\xi|, & \text{otherwise.}\end{cases}$$

For $s \in \mathbb{R}_+$, the surface energy g_{scal} is given by the asymptotic formula

$$g_{\mathrm{scal}}(s) := \lim_{T\uparrow\infty} \inf \int_{-\frac{T}{2}}^{\frac{T}{2}} \left(f(v)^2|u'|^2 + \frac{(1-v)^2}{4} + |v'|^2 \right) \mathrm{d}x,$$

where the inf is taken over all $(u,v) \in H^1([-T/2, T/2], \mathbb{R} \times [0,1])$ with

$$u(-T/2) = 0,\ u(T/2) = s, \qquad v(\pm T/2) = 1.$$

Here the surface energy is **1D**, in the sense that it involves **1D profiles** (u,v). This is due to the fact that $\mathcal{F}_\delta$ depends on ∇u through the "sliceble" potential $|\cdot|^2$.

Vectorial phase-field models: a general potential

Geometrically nonlinear framework [Conti-Focardi-F.I., in preparation]:

for $(u, v) \in H^1(\Omega, \mathbb{R}^m \times [0,1])$, let

$$\mathcal{F}_\delta(u, v) := \int_\Omega \left(f_\delta(v)^2 \, \Psi(\nabla u) + \frac{(1-v)^2}{4\delta} + \delta |\nabla v|^2 \right) dx,$$

where $\Psi \colon \mathbb{R}^{m\times d} \to [0, \infty)$ is continuous and satisfies

$$\left(\frac{1}{c}|\xi|^2 - c\right) \vee 0 \leq \Psi(\xi) \leq c(|\xi|^2 + 1), \quad \text{for all } \xi \in \mathbb{R}^{m\times d}.$$

Recall that

$$f_\delta(v)^2 := 1 \wedge \frac{\delta v^2}{(1-v)^2} \qquad (\text{with } A = B = C = 1).$$

Convergence to the sharp model

Theorem (Conti-Focardi-F.I. in preparation). It holds

$$\Gamma(L^1)\text{-}\lim_{\delta\to 0}\mathcal{F}_\delta(u,v)=\mathcal{F}(u,v),\quad (u,v)\in L^1(\Omega,\mathbb{R}^{m+1}),$$

where

$$\mathcal{F}(u,v):=\int_\Omega h^{\mathrm{qc}}(\nabla u)\,\mathrm{d}x+\int_{J_u} g([u],\nu_u)\,\mathrm{d}\mathcal{H}^{d-1}+\int_\Omega h^{\mathrm{qc},\infty}(\mathrm{d}D^c u),$$

if $u\in(GBV\cap L^1(\Omega))^m$ and $v=1$ a.e..

Here,

$$h:=\Psi\wedge\Psi^{1/2},$$

$$h^{\mathrm{qc}}(\xi):=\inf\{\int_{(0,1)^d} h(\xi+\nabla\varphi)\mathrm{d}x:\ \varphi\in C_c^\infty((0,1)^d,\mathbb{R}^m)\}$$

$$h^{\mathrm{qc},\infty}(\xi):=\limsup_{t\to\infty}\frac{h^{\mathrm{qc}}(t\xi)}{t}.$$

The limit energy g **involves vectorial profiles**: for $(z,\nu) \in \mathbb{R}^m \times S^{d-1}$

$$g(z,\nu) := \lim_{T\to\infty} \inf \frac{1}{T^{d-1}} \mathcal{F}_1^\infty(u,v,Q_T^\nu)$$

$$\mathcal{F}_1^\infty(u,v,Q_T^\nu) := \int_{Q_T^\nu} \Big(f(v)^2 \, \Psi_\infty(\nabla u) + \frac{(1-v)^2}{4} + |\nabla v|^2 \Big) dx$$

where the inf is taken over all $(u,v) \in H^1(Q_T^\nu, \mathbb{R}^m \times [0,1])$ with

$$u = (z\chi_{\{x\cdot\nu>0\}}) * \varphi, \qquad v = \chi_{\{|x\cdot\nu|\geq 2\}} * \varphi, \qquad \text{on } \partial Q_T^\nu.$$

Here φ is a mollifier and $\Psi_\infty(\xi) := \lim_{t\to\infty} \frac{\Psi(t\xi)}{t^2}$ (assumed uniformly).

[Related results: Focardi '01, Alicandro-Focardi '02]

Properties of $g(z,\nu)$

For some $c > 0$ and all $z, z_1, z_2 \in \mathbb{R}^m$, $\nu \in S^{d-1}$, it holds

- optimal profiles are vectorial and satisfy

$$(u,v) = (z\chi_{\{x\cdot\nu>0\}}, 1), \qquad \text{on} \quad \{x\cdot\nu = \pm T/2\}$$

- optimal profiles are periodic in the directions orthogonal to ν
- growth:

$$\frac{1}{c}(|z| \wedge 1) \le g(z,\nu) \le c(|z| \wedge 1)$$

- subadditivity:

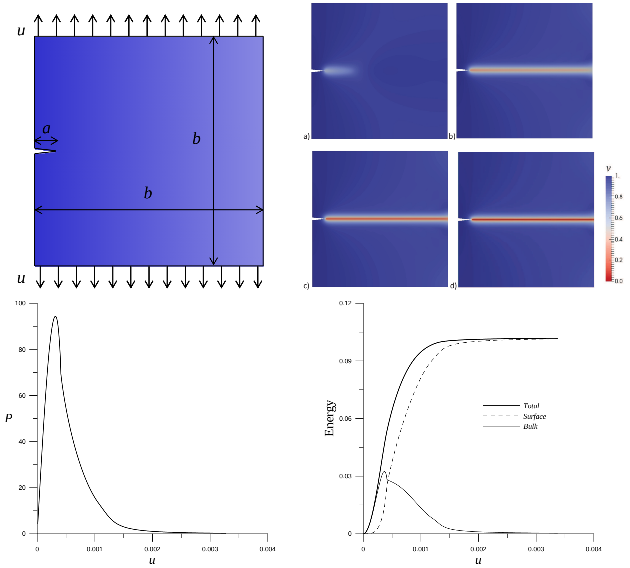
$$g(z_1 + z_2, \nu) \le g(z_1,\nu) + g(z_2,\nu)$$

- regularity:

$$g \in C^0(\mathbb{R}^m \times S^{d-1})$$

.

Numerical experiments in 2D [Freddi-F.I. '16]



The time-dependent problem

Let the boundary data evolve with the time. Of course the energy

$$F(u) := \int_{\Omega \setminus J_u} |\nabla u|^2 dx + \int_{J_u} g_{\mathrm{scal}}(|[u]|) d\mathcal{H}^{d-1},$$

does not depend on time, it is completely static. The dependence on time is related to the choice of the irreversibility condition.

(Hard) problems:

- to construct a quasi-static evolution for the cohesive sharp model without prescribing the crack path;
- to prove a phase-field approximation of such evolution.

Time-dependence in the brittle case

Let u_0 [resp. (u_0, v_0)] be a known minimum point at time t_0 under some b.c.. How to define u [resp. (u, v)] at time $t > t_0$ under a new b.c.?

- brittle sharp (Mumford-Shah) case: $\Gamma_0 := J_{u_0}$ and

$$\text{take } u \text{ minimizing:} \qquad z \mapsto \int_\Omega |\nabla z|^2 dx + \mathcal{H}^{d-1}(J_z \setminus \Gamma_0), \qquad (1)$$

  $\Gamma := J_u \cup \Gamma_0,$
  construction of quasistatic ev.: [Dal Maso-Toader '02, Chambolle '03, Francfort-Larsen '03, Dal Maso-Francfort-Toader '05, Knees-Mielke-Zanini '08, Dal Maso-Lazzaroni '10, Lazzaroni '11, Friedrich- Solombrino '17,...]

- brittle phase-field (Ambrosio-Tortorelli) case:

$$\text{take } (u, v) \text{ minimizing:} \qquad (z, w) \mapsto F_\delta^{AT}(z, w), \qquad w \leq v_0; \qquad (2)$$

  construction + convergence to sharp as $\delta \to 0$: [Giacomini '05].

Time-dependence in the cohesive case

Let u_0 [resp. (u_0, v_0)] be a minimum point at time t_0 under some b.c..
How to define u [resp. (u, v)] at time $t > t_0$ under a new b.c.?

- cohesive sharp case with prescribed crack path Γ: take (u, α) min:

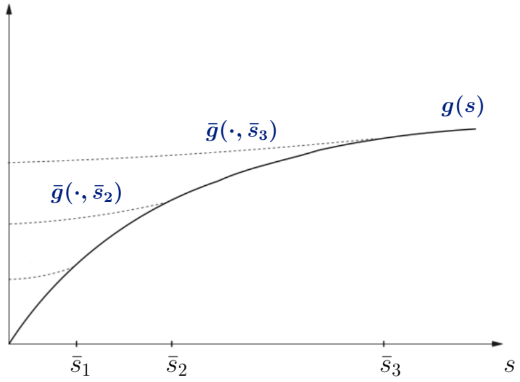
$$(z, \beta) \mapsto F(z) + \int_\Gamma \bar{g}(|[z]|, \beta) d\mathcal{H}^{d-1}, \quad \beta \geq |[u_0]|. \tag{3}$$

  Several choices for $\bar{g}$: [Dal Maso-Zanini '07, Cagnetti-Toader '11]; see also [Larsen-Slastikov '14, Larsen-Li '16, Almi '17, Artina-Cagnetti-Fornasier-Solombrino '17, Negri-Scala '17 & '20, Thomas-Zanini '17, Negri-Vitali '18, Crismale-Lazzaroni-Orlando '18,. . .].

- cohesive phase-field (Conti-Focardi-F.I.) case: we could keep $w \leq v_0$, but how to relate it to a $\bar{g}$ in the limit? [still open]

Quasi-static evolution in 1D: surface energy

[Bonacini-Conti-F.I. '21]: 1D and crack not prescribed.
The energy spent during the loading might be partially recovered during the unloading.



Quasi-static evolution in 1D: surface energy

Let $\bar{g}\colon [0,+\infty)\times[0,+\infty)\to[0,+\infty)$ be

- loading/unloading: continuous, nondecreasing in each variable and

$$\bar{g}(s,s') = \bar{g}(s,0) \text{ if } s \geq s';$$

- behavior at ∞: $\bar{g}(s,s') \leq 1$ and $\lim_{s\to+\infty} \bar{g}(s,s') = 1$ for any s';
- behavior at 0: there exist $\ell, \tilde{\ell} > 0$ and $1 < p < 2$ such that

$$\bar{g}(s,0) = \ell s - \tilde{\ell} s^p + o(s^p) \qquad \text{as } s \to 0^+;$$

- subadditivity: for every $s_1, s_2, s' \geq 0$,

$$\bar{g}(s_1+s_2,s') \leq \bar{g}(s_1,0) + \bar{g}(s_2,s')\,;$$

  if in addition $s_1 > 0$ and $s_2 \vee s' > 0$, the inequality is strict.

As in the static case, in order to make the minimization well-posed, we have to relax functional (3). We consider a bar $\Omega := [0,1]$ in traction; for

$$b : [0, T] \to \mathbb{R}^2 \quad (=\text{boundary conditions})$$

$$\Gamma \subset [0,1] \quad (=\text{cracks at previous times})$$

$$s : \Gamma \to (\bar{s}, +\infty) \quad (=\text{maximal amplitudes at previous times})$$

$$\bar{s} > 0 \quad (=\text{reversibility threshold})$$

we would like to iteratively minimize

$$\Phi(u; \Gamma, s) := \int_0^1 h_{\text{scal}}(|u'|)\mathrm{d}x + \sum_{x \in J_u^b \cup \Gamma} \bar{g}(|[u]|, s) + \ell |D^c u|(0,1)$$

among u attaining boundary conditions. Here $h_{\text{scal}}(z) := (z^2 \wedge \ell z)^{\text{conv}}$.

The time-discrete (sharp) cohesive evolution

We construct a time-discrete evolution for Φ, corresponding to data $b \in H^1([0,T],\mathbb{R}^2)$ and penalization $w \in AC([0,T],L^\infty((0,1)))$. We fix a reversibility threshold $\bar{s} > 0$ and a time step $\tau > 0$,

$$0 = t_0 < t_1 < \cdots < t_{N_\tau+1} = T.$$

Step 0: select a solution u_0^τ of the minimum problem

$$\min\left\{F_{\text{scal}}(u) + \int_0^1 |u - w_0^\tau|^2 dx : \; u \in BV((0,1)),\; b.c.\right\}.$$

Set

$$\Gamma_0^\tau := \{x \in J_{u_0^\tau} : \; |[u_0^\tau](x)| > \bar{s}\}, \qquad s_0^\tau(x) := |[u_0^\tau](x)|.$$

Step k: assume to have constructed $u_i^\tau \in BV((0,1))$, $\Gamma_i^\tau \subset [0,1]$ (finite), and $s_i^\tau : (\bar{s}, +\infty)$ for $i \leq k-1$. Select a solution u_k^τ of the minimum problem

$$\min\left\{\Phi(u; \Gamma_{k-1}^\tau, s_{k-1}^\tau) + \int_0^1 |u - w_{k-1}^\tau|^2 dx : \ u \in BV((0,1)), \ b.c.\right\}.$$

Set

$$\Gamma_k^\tau := \Gamma_{k-1}^\tau \cup \{x \in J_{u_k^\tau} : \ |[u_k^\tau](x)| > \bar{s}\},$$

$$s_k^\tau := \begin{cases} s_{k-1}^\tau \vee |[u_k^\tau]| & \text{in } \Gamma_{k-1}^\tau, \\ |[u_k^\tau]| & \text{in } \Gamma_k^\tau \setminus \Gamma_{k-1}^\tau. \end{cases}$$

Note: it turns out that Γ_k^τ is finite ($J_{u_k^\tau}$ is not) and $u_k^\tau \in SBV((0,1))$.

The time-continuous (sharp) cohesive evolution

We want to pass to the limit as the time-step tends to 0. Fix $\tau_n \to 0$ and

$$0 = t_0^n < t_1^n < \cdots < t_{N_n+1}^n = T.$$

Let

$$t \mapsto \big(u_n(t) := u_{\tau_n}(t),\, \Gamma_n(t) := \Gamma_{\tau_n}(t),\, s_n(t) := s_{\tau_n}(t)\big), \qquad \text{for } t \in [0, T],$$

be the piecewise constant interpolation of the time-discrete evolution.

Threshold $\bar{s}$ + strict subadditivity of $\bar{g} \implies \#\Gamma_n(t) \leq c$ uniformly

Quasi-static evolution in 1D (crack not prescribed)

Theorem (Bonacini-Conti-F.I. '21) For all $t \in [0, T]$, there exist $u(t) \in BV((0,1))$, $\Gamma(t) \subset [0,1]$ finite, $s(t) : \Gamma(t) \to [\bar{s}, \infty)$, such that

- (initial condition) $u_0 := u(0)$ minimizes on $BV((0,1))$, b.c.,

$$F_{\mathrm{scal}}(v) + \|v - w(0)\|_2^2,$$

  and $\Gamma_0 = \{|[u_0]| > \bar{s}\}$, $s_0 = |[u_0]|$;

- (irreversibility) $\Gamma(t_1) \subset \Gamma(t_2)$ and $s(t_1) \leq s(t_2)$ for $t_1 \leq t_2$;

- (memory) for $t > 0$,

$$\{x \in J_{u(t)} : |[u(t)]| > \bar{s}\} \subset \Gamma(t) \subset \{|[u(t)]| \leq s(t)\};$$

Quasi-static evolution in 1D (crack not prescribed)

- (static equilibrium) for $t > 0$, $u(t)$ minimizes on $BV((0,1))$, b.c.,

$$\mathcal{E}(v,t) := \Phi(v;\Gamma(t),s(t)) + \|v - w(t)\|_2^2;$$

- (non-dissipativity) the total energy $\mathcal{E}(t) := \mathcal{E}(u(t),t)$ satisfies

$$\mathcal{E}(t) = \mathcal{E}(0) + \int_0^t \int_0^1 \Big(h'((u)')(\dot{b})' + 2(u-w)(\dot{b} - \dot{w}) \Big) dx dr.$$

Irreversibility in the phase-field model

Ambrosio-Tortorelli: $v_\delta \sim 0$ on the jump;
Conti-Focardi-F.I.: v_δ reaches a value depending on $|[u]|$ on the jump.

Key facts: for $\bar{x} \in J_u$ with amplitude $s := |[u](\bar{x})|$ we have:

- $g_{\text{scal}}(s)$ has a unique optimal profile β_s for the damage variable;
- in particular $m_s := \min_{t\in\mathbb{R}} \beta_s(t)$ is determined by the opening s;
- a recovery sequence v_δ blows up towards β_s around $\bar{x}$.

This suggests to impose irreversibility through a monotonicity constraint on the minima of v_δ.

Irreversibility in the phase field model

[Bonacini-Conti-F.I. '21]: Fix a finite set $\Gamma \subset [0,1]$ and $s' : \Gamma \to (0,\infty)$ (pre-existing crack and amplitude). We have

$$\tilde{F}_\delta(u,v) := F_\delta^{\mathrm{scal}}(u,v), \qquad \text{if } v(x) \le m_{s'(x)} \text{ on } \Gamma,$$

$$\tilde{F}_\delta(u,1) \xrightarrow{\Gamma} \Phi(u) := \int_0^1 h_{\mathrm{scal}}(|u'|)dx + \sum_{x \in \Gamma \cup J_u} \bar{g}(|[u]|, s') + |D^c u|((0,1)),$$

for $u \in BV((0,1))$, where

$$\bar{g}(s, s') := \inf \int_{-\infty}^{+\infty} \left(f^2(\tilde{v})|\tilde{u}'|^2 + \frac{(1-\tilde{v})^2}{4} + |\tilde{v}'|^2 \right) dt,$$

$$(\tilde{u}, \tilde{v}) \text{ as for } g_{\mathrm{scal}} \quad + \quad \inf \tilde{v} \le m_{s'}.$$

...thank you for your attention!